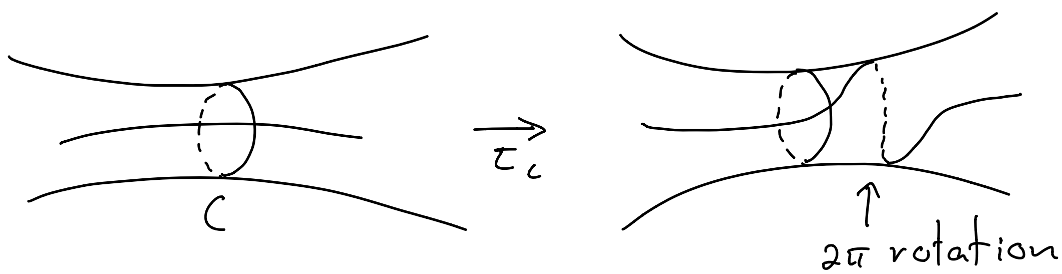
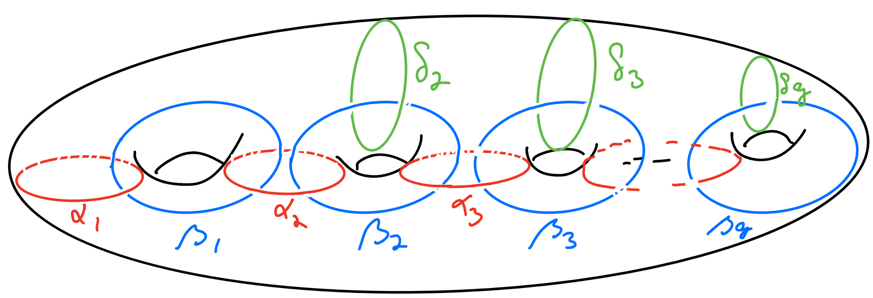
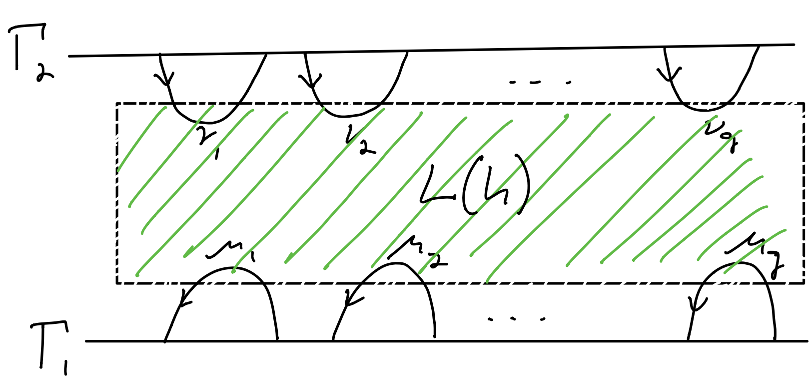
Let $C$ be a simple closed curve on $\Sigma$ and denote by $\tau_C$ the "Dehn twist" along $C$




By Lickorish, the mapping class group $\mathcal{M}_g$ is generated by isotopy classes of Dehn twists along $3g-1$ curves $\alpha_i, \beta_i$, $1 \le i \le g$, $\delta_j$, $2 \le j \le g$:




Now embed graphs $T_1$ and $T_2$ in $\mathbb{R}^2 \times I$:



Let $H_1$ and $H_2$ be handlebodies obtained as regular neighbourhoods of $T_1$ and $T_2$

For $h \in \mathcal{M}_g$, denote by $M$ 3-manifold obtained by gluing $H_1$ and $H_2$ by $h : \partial H_1 \to \partial H_2$

Consider

$$\mathbb{R}^2 \times I \subset \mathbb{R}^3 \cup \{\infty\}$$

and take link

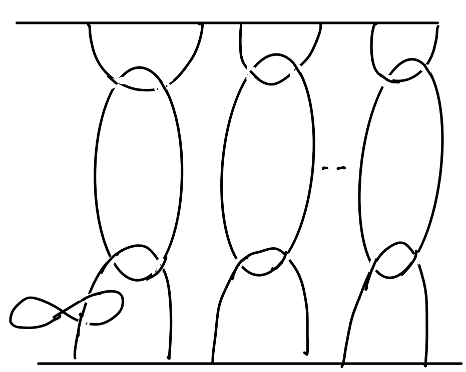
$$L \subset \mathbb{R}^2 \times I, \quad L \cap (T_1 \cup T_2) = \emptyset$$

such that Dehn surgery on $L$ gives $M$.

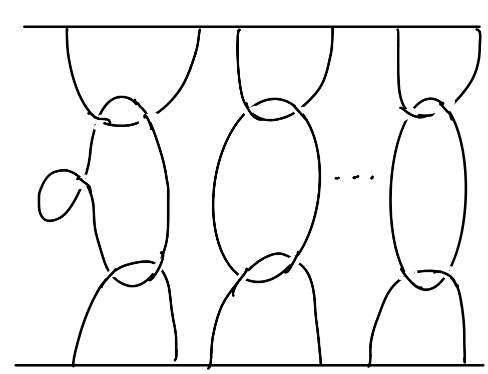
$\longrightarrow$ $L = L(h)$,

$T_1$ together with $T_2$ and $L(h)$ form a $(2g, 2g)$ framed tangle.

For example, we get for the Lickorish gener.:



$\alpha_1$



$\beta_1$

$\rightarrow$ denote the resulting $(2g, 2g)$-tangle by $T(h)$

Choose coloring $\mu_1, \dots, \mu_g$ for $T_1$ and $\nu_1, \dots, \nu_g$ for $T_2$ as shown above

Choose coloring $\lambda : \{1, 2, \dots, m\} \rightarrow P_+(k)$ for components of $L(h)$

$\rightarrow$ obtain tangle operator

$$J(T(h); \lambda)_{\mu\nu} : V_{\mu_1 \mu_1^* \cdots \mu_g \mu_g^*} \rightarrow V_{\nu_1 \nu_1^* \cdots \nu_g \nu_g^*}$$

and define

$$\rho(h)_{\mu\nu} = \sqrt{S_{0\mu_1} \cdots S_{0\mu_g}} \sqrt{S_{0\nu_1} \cdots S_{0\nu_g}}$$

$$\times\, C^{\sigma(L(h))} \sum_{\lambda} S_{0\lambda_1} \cdots S_{0\lambda_m} J(T(h); \lambda)_{\mu\nu}$$

Define

$$\rho(h) : V_\Sigma \rightarrow V_\Sigma$$

by $\rho(h) = \bigoplus_{\mu, \nu} \rho(h)_{\mu\nu}$.

$\longrightarrow$ obtain map $\rho : \mathcal{M}_g \longrightarrow GL(V_\Sigma)$

$\rho(h)$ depends only on the isotopy class of $h$ and not on the way of expressing $h$ as product of Lickorish generators.

For $x, y \in \mathcal{M}_g$, consider links $L(x), L(y)$ and $L(xy)$ and set

$$\zeta(x\,y) = C^{\sigma(xy) - \sigma(x) - \sigma(y)}$$

Then the above implies

Proposition 4:

The above map $\rho : \mathcal{M}_g \longrightarrow GL(V_\Sigma)$ satisfies

$$\rho(xy) = \zeta(x, y)\,\rho(x)\,\rho(y)$$

for any $x, y \in \mathcal{M}_g$. Thus, $\rho$ is a projectively linear rep. with 2-cocycle $\zeta$.

"2-cocycle": $\zeta(xy, z)\,\zeta(x, y) = \zeta(x, yz)\,\zeta(y, z)$

which follows from

$$\rho((xy)z) = \rho(x(yz))$$

In the case $g = 1$, $\mathcal{M}_1 \cong SL(2, \mathbb{Z})$ and $\dim V_\Sigma = K+1$ with basis $\{v_\lambda\}$

Lemma 4:

The action $\rho$ of $SL_2(\mathbb{Z})$ on basis $\{v_\lambda\}$ is given by

$$S v_\lambda = \sum_\mu S_{\lambda\mu}\, v_\mu$$

$$T v_\lambda = \exp\left(2\pi\sqrt{-1}\,\Delta_\lambda\right) v_\lambda$$

## Basis of $V_\Sigma$:

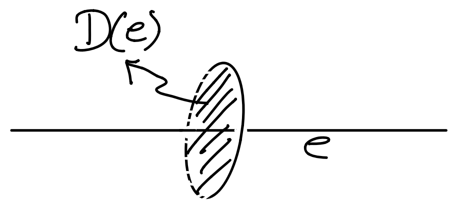
- correspond to admissible coloring of different trivalent graphs
- related by connection matrices of KZ-eq.

Take particular basis corresponding to $\Gamma'$

$\rightarrow$ neighborhood of $\Gamma'$ in $\mathbb{R}^3$ gives a handlebody $H$ of genus $g$

$\rightarrow \partial H = \Sigma$

For $e$ edge in $\Gamma'$, set




$$C(e) = \partial D(e)$$

$\longrightarrow$ Dehn twist along $C(e)$ acts diagonally on $\Gamma'$-basis

Set $\quad \lambda : \mathrm{Edge}(\Gamma') \longrightarrow P_+(K)$

$$\rightarrow \rho(\tau_{C(e)})\, v_\lambda = \exp\left(-2\pi\sqrt{-1}\,\Delta_{\lambda(e)}\right) v_\lambda$$

For example, in the previous graph, $\delta_1, \delta_2, \dots, \delta_g$ diagonalized simultaneously

## Witten's invariants:

Let $M$ be closed oriented 3-mfd. obtained by gluing two handle bodies:

$$M = H_g \cup_h (-H_g),$$

$$h : \partial H_g \to \partial H_g, \quad h \in \mathcal{M}_g$$

"Heegaard splitting"

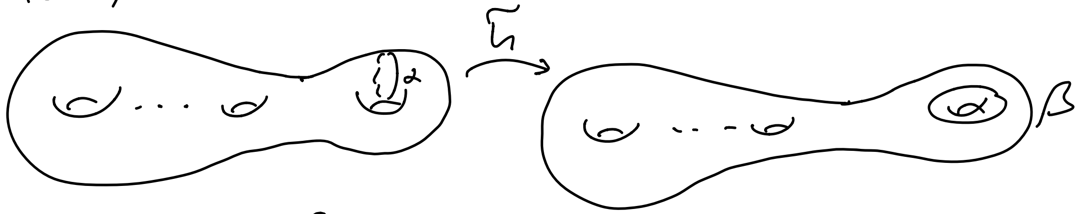
## Proposition 5:

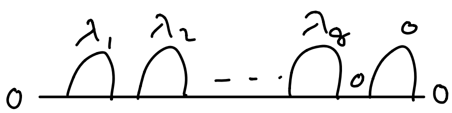
Let $M = H_g \cup_h (-H_g)$ be a Heegaard splitting of a closed oriented 3-manifold $M$. Then, the Chern-Simons partition function of $M$ is

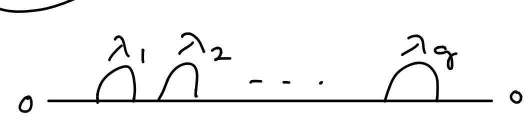
$$Z_k(M) = S_{00}^{-g+1} \langle v_0^* | \rho(h) v_0 \rangle$$

where $v_0^*$ is the dual element of $v_0$ in $V_\Sigma^*$ and $\langle \, | \, \rangle$ is the canonical pairing between $V_\Sigma^*$ and $V_\Sigma$.

Now, add a 1-handle to $H_g$ to obtain $H_{g+1}$:




$\tilde{v}_\lambda =$ 


$v_\lambda =$ 

Extend $h \in \mathcal{M}_g$ to $\tilde{h} \in \mathcal{M}_{g+1}$ so that

$\tilde{h}(\alpha) = \beta \longrightarrow H_{g+1} \cup_{\tilde{h}} (-H_{g+1})$ is connected sum of $M = H_g \cup_h (-H_g)$ and $S^3$

$\rightarrow$ homeomorphic to $M$

"elementary stabilization"

two Heegaard splittings $H_g \cup_h (-H_g)$ and $H_g \cup_{h'} (-H_g)$ are equivalent if

$$h' = h_1 \circ h \circ h_2, \quad h_1, h_2 \in \mathcal{M}_g$$

(extended to $H_g$)

$$\longrightarrow \quad S_{00}^{-g+1} \langle v_0^* | \rho(h) v_0 \rangle = S_{00}^{-g'+1} \langle v_0^* | \rho(h') v_0 \rangle$$

follows from

$$\langle v_0^* | \rho(\tilde{h}) v_0 \rangle = S_{00} \langle v_0^* | \rho(h) v_0 \rangle$$